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REPORT

Summary

By a theorem of Hanna Neumann, an amalgam of an arbitrary collection of groups is embeddable if and only if the reduced amalgam of such groups is embeddable. So, in order to discuss the embeddability of an amalgam of finite groups in a finite groups, we first consider the embeddability of the reduced amalgam in a finite group. The general problem has avoided solution for the last thirty years. In this report we have investigated the embeddability of an amalgam of three finite dihedral groups in a finite group. The groups are given in the form

$$A = \langle a, b : a^2 = b^2 = (ab)^{\ell} = 1 \rangle$$

$$B = \langle b, c : b^2 = c^2 = (bc)^{m} = 1 \rangle$$

$$C = \langle c, a : c^2 = a^2 = (ca)^{n} = 1 \rangle$$

The amalgam $\underline{\underline{A}}$ formed by these groups is their reduced amalgam. If any two of the ℓ, m, n, say ℓ, m are equal to 2 then

$$G = C \times \langle b : b^2 = 1 \rangle$$

embeds the amalgam and is finite. Here we examine the problem in more generality.

Detailed Report:

The group

$$F = \langle a,b,c = a^2 = b^2 = c^2 = (ab)^{\ell} = (bc)^{m} = (ca)^{n} = 1\rangle$$

described by Coxeter and Moser as the group of reflections in the sides of a spherical triangle with angles π/ℓ, π/m, π/n *was proved to be the generalised free product of the groups*

$$A = \langle a,b : a^2 = b^2 = (ab)^{\ell} = 1\rangle$$

$$B = \langle b,c : b^2 = c^2 = (bc)^{m} = 1\rangle$$

$$C = \langle c,a : c^2 = a^2 = (ca)^{n} = 1\rangle$$

in ( ) *in F, is known to be finite if* $\frac{1}{\ell} + \frac{1}{m} + \frac{1}{n} > 1$ *and infinite otherwise. We can also write F as*

$$F = \langle g,h,c : g^{m} = h^{n} = (gh)^{\ell} = c^2 = (gc)^2 = (ch)^2 = 1\rangle$$

It is easy to note that F is then a split extension of

$$P = (m,n,\ell) = \langle g,h = g^{m} = h^{n} = (gh)^{\ell} = 1\rangle$$

by a cyclic group of order 2. P belongs to the well known family of groups called 'polyhedral groups' which is interesting in the sense that many of the known finite simple groups are factor groups of this group.

We require the following definitions and concepts.

Let $\{G_\alpha : \alpha \in \Lambda\}$ *be a collection of groups with* $G_\alpha \cap G_\beta = H_{\alpha\beta}$.

The amalgam of G_α, $\alpha \in \Lambda$ *is an 'incomplete group' whose elements are those of* G_α *with the elements of* $H_{\alpha\beta}$ *thought* of as identified in the two groups G_α, G_β, $\alpha, \beta \epsilon \Lambda$. If there is a group G containing all $G_\alpha, \alpha \epsilon \Lambda$ such that in G, G_α and G_β intersect precisely in a subgroup $H_{\alpha\beta}$, $\alpha, \beta \epsilon \Lambda$, then G is said to embed the amalgam of G_α. If G_α, $\alpha \epsilon \Lambda$, are all finite, then the amalgam formed by these G_α is said to be a finite amalgam. If a finite group G exists which contains all $G_\alpha, \alpha \epsilon \Lambda$ with their correct intersections $H_{\alpha\beta} = G_\alpha \cap G_\beta, \alpha, \beta \epsilon \Lambda$, then we say that the amalgam $\underline{\underline{A}}$ is embeddable in finite group.

Given an amalgam $\underline{\underline{A}}$ of group G_α, it is, in general, not true that $\underline{\underline{A}}$ be embeddable in a group much less that it may be embeddable in a finite group. It is, of course, known that amalgam of two groups is always embeddable and, in fact, embeddable in a finite group, by a well known result of B.H.Neumann ( ). The embeddability problem for three or more groups, however, is extremely difficult. Even for the case of an amalgam of three finite groups, no necessary and sufficient conditions for such an amalgam to have a finite embedding are known.

The finite amalgam considered here is an amalgam of three finite dihedral groups given in the form

$$A = \langle a, b = a^2 = b^2 = (ab)^\ell = 1 \rangle$$
$$B = \langle b, c = b^2 = c^2 = (bc)^m = 1 \rangle$$
$$C = \langle c, a = c^2 = a^2 = (ca)^n = 1 \rangle$$

The embeddability of this amalgam was established in ( ). The finite embeddability of this amalgam is discussed in the following paragraphs.

Consider the finite amalgams:

$$\underset{=}{A}_1 = am(A,B : \langle b\rangle), \quad \underset{=}{A}_2 = am(B,C : \langle c\rangle)$$

and

$$\underset{=}{A}_3 = am(C,A : \langle a\rangle).$$

These are three finite amalgams and are embeddable in finite groups G_1, G_2 and G_3 respectively. Since G_1 is generated by a,b,c, the pairs a, b:b,c generate in G_1 groups isomorphic to A,B and there intersection is precisely <b> . However the subgroup of G_1 generated by a,c, may, in general, be different form C but is still a dihedral group. Similarly for the groups G_2 and G_3.

Let us now consider the ordinary free product:

$$F = \langle a\rangle * \langle b\rangle * \langle c\rangle$$
$$= \langle a,b,c : a^2 = b^2 = c^2 = 1\rangle$$

of the cyclic groups $\langle a = a^2 = 1\rangle$, $\langle b : b^2 = 1\rangle$, $\langle c : c^2 = 1\rangle$, $i = 1,2,3$.

Since each G_i, $i = 1,2,3$, is also generated by a,b,c; by the property of free products, all these are isomorphic to factor groups of F. Thus there are normal subgroups N_1, N_2, N_3 such that

$$N \cap \langle a : a^2 = 1\rangle = \{1\},$$
$$N \cap \langle b : b^2 = 1\rangle = \{1\},$$
$$N \cap \langle c : c^2 = 1\rangle = \{1\},$$

Since N_1, N_2, N_3 all are normal subgroups of F and have finite index in F, N is a normal subgroup of F and has finite index in F. Thus F/N a finite group.

In F/N, the groups generated by the pairs aN, bN; bN, cN; cN, aN are again dihedral groups say, A_1, B_1, C_1 respectively. These may again be not isomorphic with A, B and C respectively. Let the orders of A_1, B_1, C_1 respectively be $2\ell_1$, $2m_1$, and $2n_1$/ Then ℓ_1, m_1, n_1 are divisible by ℓ, m and n respectively so that A, B and C are homomorphic images of A_1, B_1, C_1 respectively.

Consider now the normal closures:

$$X = \langle f(ab)^{\ell} f^{-1} N : f \in F \rangle$$

$$Y = \langle g(bc)^{m} g^{-1} N : g \in F \rangle$$

$$Z = \langle h(ca)^{n} h^{-1} N : h \in F \rangle$$

of the cyclic groups $\langle (ab)^{\ell} N \rangle$, $\langle (bc)^{m} N \rangle$, $\langle (ca)^{n} N \rangle$ in F/N respectively. Since

$$\langle f(ab)^{\ell} f^{-1} : f \in N \rangle$$

$$\langle g(bc)^{m} g^{-1} : g \in F \rangle$$

$$\langle h(ca)^{n} h^{-1} : h \in F \rangle$$

are normal in F; X, Y, Z are normal in F/N. Let

$$Q = \langle X, Y, Z \rangle$$

be the subgroup of F/N generated by X, Y and Z. Then Q is a normal subgroup of F/N. Also since N_1, N_2, N_3 are normal in F,

$$X \subseteq N_1 \cap N_3, \quad Y \subseteq N_1 \cap N_2, \quad Z \subseteq N_2 \cap N_3$$

so that

$$xyx^{-1}y^{-1} \in N_1 \cap N_2, \quad x = f(cb)^{\ell} f^{-1}, \; y = g(bc)^{m} g^{-1}$$

Also since $Z \subseteq N_2 \cap N_3$, $XY \cap Z \subseteq N_2 \cap N_3$. But X and Y are contained in N_1. As N_1 is a subgroup, $XY \subseteq N_1$. Thus $XY \cap Z \subseteq N_1$. So $XY \cap Z \subset N$. Similarly $YZ \cap X$ and ZX Y are contained in N. Hence O is the direct product of X,Y,Z.

We claim that the factor group of F/N by O embeds the amalgam of A,B,C. For this we have to show that

(i) O contains no elements of the form

$$a^{\varepsilon}(ab)^{i}N,\ b^{\delta}(bc)^{j}N,\ c^{\omega}(ca)^{k}N$$

$\varepsilon=0$ or 1, $\delta=0$ or 1, $\omega=0$ or 1, $0\leq i<\ell$, $0\leq j<m$, $0\leq k<n$

so that the factor group of F/N by O contains isomorphic copies of A,B,C.

(ii) O contains no element of the form

$$a^{\varepsilon}(ab)^{i}.\ b^{\delta}(bc)^{j}N,\ b^{\delta}(bc)^{i}.\ c^{\omega}(ca)^{k}N,\ c^{\omega}(ca)^{k}.a^{\omega}(ab)^{i}N$$

when $\varepsilon=0$, or 1, $\delta=0$ or 1, $\omega=0$ or 1, $0\leq i<\ell$, $0\leq j<m$, $0\leq k<n$ so that in the corresponding factor group, $A,B;B,C;$ and C,A have their correct intersections.

For (i) suppose that

$$a^{\varepsilon}(ab)^{i}N \ \varepsilon\ O$$

Then

$$a^{\varepsilon}(ab)^{i} = \prod_{\alpha=1}^{p} f_{\alpha}(ab)^{k}f_{\alpha}^{-1} \ .\ \prod_{\beta=1}^{q} g_{\beta}(bc)^{m}g_{\beta}^{-1} \ .\ \prod_{\gamma=1}^{r} h_{\gamma}(ca)^{n}h_{\gamma}^{-1}\, n \qquad (\ )$$

$n\ \varepsilon\ N$. Since F is the free product of three 2-cycles generated by a,b,c respectively, each element of F has a unique normal form and the number of factors, after reduction by cancellation or amalgamation, is uniquely determined. Hence (1) gives two representations for one and the same element of F. Consider the right hand side of (1). Since the left hand side of (1) does not contain any factor involving c, the right hand side of (1) also does not contain any c as a factor. Thus the product.

$$\Pi g_\beta (bc)^m g_\beta^{-1} \ . \ \prod_{\gamma=1}^{r} h_\gamma (ca)^n h_\gamma^{-1}$$

does not occur in (1). Also since $n \in N \subseteq N_1$, $\prod_\beta^{p} \cdot f_\alpha (ab)^{\ell} f_\alpha^{-1} \in N_1$, we find that the right hand side of (1) is contained in N_1.

Hence

$$a^\varepsilon (ab)^i \in N_1$$

But then F/N_1 does not embed the amalgam of the groups A and B amalgamating $\langle b : b^2 = 1\rangle$, because then A collapses in F/N, a contradiction. Hence $a^\varepsilon (ab)^i N \notin Q$. Similarly, $b^\delta (bc)^j N \notin Q$, $c^\omega (ca)^h N \notin Q$ and we have (1).

To see that condition (ii) is satisfied, let

$$a^\varepsilon (ab)^i \ . \ b^\delta (bc)^j N \in Q.$$

Then again,

$$a^\epsilon (ab)^i . b^\delta (bc)^j = \prod_{\alpha=1}^{p} f_\alpha (ab)^{\ell} f_\alpha^{-1} \cdot \prod_{\beta=1}^{q} g_\beta (bc)^m g_\beta^{-1} \cdot \prod_{\gamma=1}^{r} h(ca)^n h_\gamma^{-1} n,$$

$n \in N$. Once more, sinde (2) gives two expression for the normal form of an element in F and the left hand side does not contain any factor of the form $(ca)^n$,

$$\prod_{\gamma=1}^{r} h_\gamma (ca)^n h_\gamma^{-1}$$

does not appear in (2). But then, since $XY \quad N_1, N \quad N$

$$\prod_{\alpha=1}^{p} f_\alpha (ab)^{\ell} f_\alpha^{-1} \ . \ \prod_{\beta=1}^{q} g_\beta (bc)^{m} g_\beta^{-1} \quad n$$

is in N_1 so that

$$a^{\varepsilon}(ab)^{i} \ b^{\delta}(bc)^{j} \ \varepsilon \ N_1 .$$

But this again is impossible because F/N_1 embeds the amalgam of A,B amalgamating $\langle b : b^2 = 1 \rangle$. Thus

$$a^{\varepsilon}(ab)^{i} . b^{\delta}(bc)^{j} N \not\in Q .$$

Similarly for the remaining two expressions.

Consequently the factor group of F/N by Q contains isomorphic copies of A,B,C with their correct intersections and so embeds their **amalgam**. Being a facto group of a finite group, this is the required finite embedding of amalgam of A,B and C.

From the above discussion one can notice how difficult the general problem of embeddability of a finite amalgam in a finite group is. We do not even how as to when an amalgam of 3 or more groups is embeddable in a group, much less its being embeddabl in a finite group. Further work on this problem is continuing.

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