Numerical Approximation of Rapidly Oscillatory Bessel Integral Transforms

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Abstract

We present a new procedure of Levin type which is based on Gaussian radial basis function for evaluation of rapidly oscillating integrals that contains first kind of the Bessel function $J_v(\omega x)$. Multi-resolution quadrature rules like hybrid and Haar functions are used in the context of Bessel oscillatory integrals as well. Numerical test problems are solved to verify the accuracy and efficiency of the new methods.

Keywords: Rapidly oscillatory integrand, Bessel Function of the first kind, Gaussian RBF, Hybrid and Haar functions.

INTRODUCTION

Bessel oscillatory integrals have applications in many areas of science and engineering such as astronomy, optics, electromagnetic, seismology, image processing etc.[1, 2]. In the present paper, we have evaluated the Bessel type of oscillatory integrals of the form:

$$I[g,\kappa] = \int_c^d g(x) J_v(\omega x) dx, \tag{1}$$

Where g (x) is non-oscillatory smooth function and $J_v(\omega x)$is the first kind of Bessel function with v>0, order of the Bessel function.

Many accurate methods have been developed for numerical evaluation of the integrals of the form (1) such as Levin collocation method [3-5], generalized quadrature rule [6-9], Homotopy perturbation method [1, 10] and some more. Levin [3] proposed a new approach for numerical evaluation of Bessel type of oscillatory integrals of the form (1). In the same paper, he extended the method to the solution of integrals with Bessel-trigonometric and square of the Bessel oscillatory integrands. In the next paper [4], Levin calculated some theoretical error bounds for the method given in [3].

Xiang [5] investigated some new theoretical error bounds for the method reported in [3] with asymptotic order of convergence $O\left(\kappa^{-\frac{5}{2}}\right)$.

In this paper, we have used collocation with Gaussian RBF as basis function (GRBF) instead of monomials [3, 5]. The asymptotic order of convergence of the proposed method GRBF is $O\left(\kappa^{-\frac{7}{2}}\right)$. Multi-resolution quadrature rules like hybrid and Haar functions [11] are used for evaluation of the integrals (1) as well.

GAUSSIAN RBF BASED QUADRATURE

According to this procedure, a new technique is proposed to evaluate a class of oscillatory integrals of the form:

$$\int_c^d \sum_{j=1}^{n} g(x)\, Y(\kappa, x) dx = \int_c^d G(x).\boldsymbol{Y}(\kappa, x) dx, \quad 0 \leq c \leq x \leq d, \tag{2}$$

Where G(x) and $\boldsymbol{Y}(\kappa, x)$are vectors of the non-oscillatory and the oscillatory functions respectively. The derivative of $\boldsymbol{Y}(\kappa, x)$ is$\boldsymbol{Y}'(\kappa, x) = B(\kappa, x)\boldsymbol{Y}(\kappa, x)$, where $B(\kappa, x)$ is $n \times n$ matrix of non-oscillatory functions.

An approximate solution $\tilde{S}(x) = \sum_{i=1}^{m} w_i^{[j]} \varphi_i(x), i = 1,2, \dots, n$ is supposed to satisfy the following ODE:

$$£[S(x)] = G(x), \qquad 0 \leq c \leq x \leq d, \tag{3}$$

Where

$$£[S(x)] = S'(x) + B(\kappa, x)\, S(x).$$

Then the unknown coefficients $w_i^{[j]}$, $i = 1,2, \dots, m, j = 1,2, \dots, n$ can be determined by the interpolation condition:

$$£\left[\tilde{S}(x_k)\right] = G(x_k), \qquad k = 1,2, \dots, m. \tag{4}$$

Thus integral (1) can be evaluated as:

$$GRBF = \int_c^d [\tilde{S}'(x) + B(\kappa, x)\tilde{S}(x)].\boldsymbol{Y}(\kappa, x)\, dx$$

$$= \int_c^d \left[\tilde{S}(x).\boldsymbol{Y}(\kappa, x)\right]' dx$$

$$= \tilde{S}(d).\boldsymbol{Y}(\kappa, d) - \tilde{S}(c).\boldsymbol{Y}(\kappa, c).$$

Particularly, to compute the integral $I[f, \kappa] = \int_c^d g(x) J_v(\omega x) dx$, we assume

$$B(\kappa,x) = \begin{pmatrix} \frac{v-1}{x} & -\kappa \\ \kappa & \frac{-v}{x} \end{pmatrix}, \quad \boldsymbol{Y}(\kappa,x) = \begin{pmatrix} J_{v-1}(x) \\ J_v(x) \end{pmatrix} \text{ and } \quad \boldsymbol{G}(x) = \begin{pmatrix} 0 \\ g(x) \end{pmatrix}.$$

In this case, the approximate solution $\tilde{S}(x) = \sum_{i=1}^{m} w_i^{[j]} \varphi_i(x), i = 1{,}2$ is supposed to satisfy the ODE (4) and then we can find the values of the unknown coefficients $w_i^{[j]}$, $j = 1{,}2$, $i = 1{,}2,\dots,m$.

On substituting the values of $B(\kappa,x)$, $S(x)$ and $\boldsymbol{G}(x)$ in (4), we obtain a system of coupled equations;

$$\begin{aligned} &[\varphi'(x) + \frac{v-1}{x}\varphi(x)]\, w^{[1]} + \kappa\varphi(x)\, w^{[2]} = 0 \\ &-\kappa\varphi(x) w^{[1]} + [\varphi'(x) - \frac{v}{x}\varphi(x)]\, w^{[2]} = g(x). \end{aligned} \tag{5}$$

The (5) can then be written in matrix form as:

$$\boldsymbol{A\beta} = \boldsymbol{G},$$

Where

$$\boldsymbol{\beta} = \begin{pmatrix} w^{[1]} \\ w^{[2]} \end{pmatrix}, \quad \boldsymbol{G}(x) = \begin{pmatrix} 0 \\ g(x) \end{pmatrix},$$

And **A** is a $2m \times 2m$ square matrix. $\boldsymbol{\beta}$ and $\boldsymbol{G}(x)$ are the column matrices of order $2m \times 1$. An accurate approximate solution of the ODE (3) is the aim of this paper. For this purpose, Gaussian RBF$\varphi(r,c)$is used as basis function and is defined by;

$$\varphi(r,c) = e^{-\frac{r^2}{c^2}}, \tag{6}$$

And

$$\varphi'(r,c) = \frac{-2r}{c^2} e^{-\frac{r^2}{c^2}},$$

Where c is the shape parameter of the Gaussian RBF and $r = \sqrt{(x - xc)^2}$, xc,s are the centers of Gaussian RBF. The accuracy as well as the condition number of the system (5) depends upon the value of c. Therefore, an optimal value of the shape parameter is still an open problem. In this paper, an algorithm [12] is used for an optimal value of c. In this algorithm, the value of c is changing with change in the nodal points as well as the frequency parameter κ. In this paper, we have used $c_L = 0$ and $c_U = 3$ for finding c in the algorithm.

Solving the system of equation (5) for the unknown coefficient matrix $\boldsymbol{\beta}$ and find the approximate solution $\tilde{S}(x)$.

QUADRATURE BASED ON HYBRID AND HAAR FUNCTIONS

In this section, multi-resolution quadrature rules like hybrid functions (HFQ) and Haar wavelets (HWQ) are briefly described. The detail description and proofs of the formulae for HFQ and HWQ are given in [11, 13] .

In this paper we have used hybrid function based quadrature of order m = 8 (HF Q8) for evaluating the integral of the form: $I\,[f\,] = \int_a^b f(x)dx$. Formula for the HFQ8 is given by

$$\text{HFQ8} = \frac{8h}{1935360}\sum_{k=1}^{N}[295627\,f\left(a+\frac{h}{2}(16k-15)\right)+\ 71329\,f\left(a+\frac{h}{2}(16k-13)\right)+ 471771\,f\left(a+\frac{h}{2}(16k-11)\right)+128953\,f\left(a+\frac{h}{2}(16k-9)\right)+ 128953\,f\left(a+\frac{h}{2}(16k-7)\right)+\ 471771\,f\left(a+\frac{h}{2}(16k-5)\right)+ 71329\,f\left(a+\frac{h}{2}(16k-3)\right)+\ 295627\,f\left(a+\frac{h}{2}(16k-1)\right), \qquad (7)$$

Where $h = \frac{b-a}{8n}$.

Similarly, the formula of Haar wavelet based quadrature for computing the integral $I\,[f\,] = \int_a^b f(x)dx$ is given by HWQ $= h\ \sum_{k=1}^{N} f(x_k)$

$$= h\sum_{k=1}^{N} f(a+h(k-0.5)), \qquad (8)$$

Where $h = \frac{b-a}{2M}$ and $N = 2M$.

Note: In case of Bessel type of oscillatory integrals, we take $f(x) = g(x)\,J_v(\omega x)$.

CONVERGENCE

Some theoretical results for convergence of the proposed methods are calculated. First, we consider the error bounds of the multi resolution methods HFQ and HWQ:

Quadrature based on hybrid and Haar functions

If a = 0, b = 1, n = 4 and $h = \frac{b-a}{8n}$, then the error bound of formula (7) is calculated as:

$$|Error| = \frac{h^9}{4.54} f^{(8)}(\xi), \tag{9}$$

Where $\xi \in [a,b]$.

Similarly, for the integral $I[f] = \int_a^b f(x)dx$ and 2M = 4, then the error bound for the HWQ is defined as:

$$|Error| = \frac{h^3}{6} f''(\eta), \tag{10}$$

where $\eta \in [a,b]$.

Gaussian RBF based quadrature

Theorem 1. Let $B(\kappa,x) = (\frac{1}{\kappa}A(\kappa,x))^{-1}$ exists and $G(x), A(\kappa,x),\ \boldsymbol{\Upsilon}(\kappa,x) \in C^m[a,b]$. Also $B(\kappa,x),\ B^{-1}(\kappa,x)$ and their 2m derivatives are uniformly bounded on [a, b], then the error bound for computing the integral (1) by the Gaussian RBF based quadrature rule is given by

$$E(\kappa) = |I(g) - GRBF| = O\left(\frac{h^{m-2}}{\kappa^{7/2}}\right).$$

NUMERICAL RESULTS

In this section, some test cases are considered to verify the accuracy and efficiency of the proposed methods. The real solution of the test problem is obtained from MAPLE 15 [14]. Results in terms of absolute errors (Error) are obtained.

Test problem 1. Consider the computation of the integral [1]

$$I_1[f,\kappa] = \int_1^2 \frac{1}{1+x^2} J_v(\omega x)dx,$$

by the Gaussian RBF based quadrature and multi-resolution quadrature rules like HFQ and HWQ. Numerical results related to the frequency parameter obtained from the three methods are shown in Figure 1. The proposed method GRBF improves the accuracy as the frequency κ is increasing, while the multi-resolution methods HFQ and HWQ fail to retain the desired accuracy as shown in Figure 1. The multi resolution methods give the desired accuracy at finer nodes which becomes computationally extensive as shown in Figure 2. From both the figures, it is clear that the new method, GRBF retains the desired accuracy for high frequencies at coarser grid points.

It is clear from the Figure 3 that the new method GRBF is accurate with asymptotic order of convergence $O\left(\kappa^{-\frac{7}{2}}\right)$ at small nodes i.e. m = 10. The oscillatory behavior of the integrand is shown in Figure 4 for $\kappa = 1000$. In last, the new method is tested for high frequency value.

According to the results in Table 1, it is evident that the method GRBF is accurate and efficient at small number of nodal points.

Table 1: Absolute error and CPU time (in parenthesis) produced by the GRBF

κ	m = 10	m = 20	m = 30
10^2	$1:2418e^{-11}$ (0.0209s)	$1:5043e^{-011}$ (0:0981s)	$8:2876e^{-012}$ (0:1539s)
10^3	$3:5427\ e^{-13}$ (0:0316s)	$3:2348e^{-012}$ (0:0917s)	$4:2022e^{-014}$ (0:2088s)
10^4	$1:1548e^{-15}$ (0:0267s)	$3:9864e^{-016}$ (0:1022s)	$2:8363e^{-016}$ (0:1742s)
10^6	$3:5506e^{-19}$ (0:0404s)	$7:3052e^{-020}$ (0:0969s)	$2:8113e^{-019}$ (0:1289s)

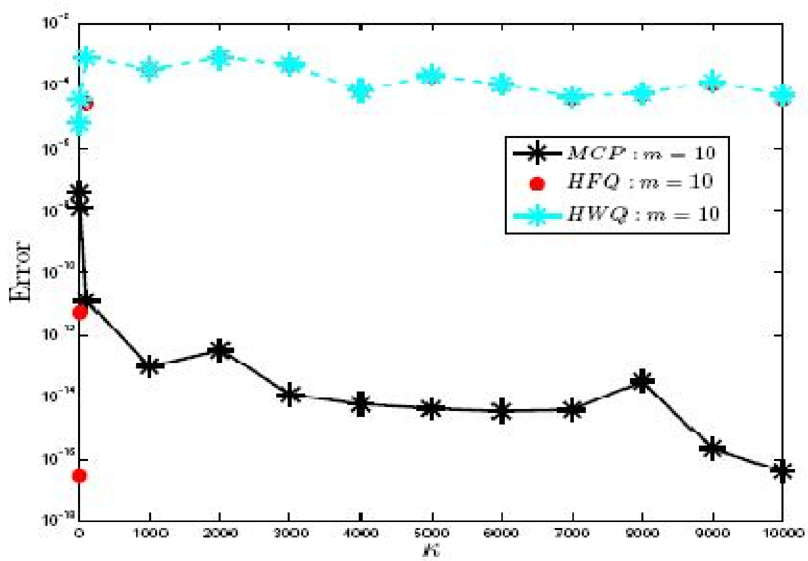

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Figure 1: Absolute error of HFQ, HWQ and GRBF for m=10

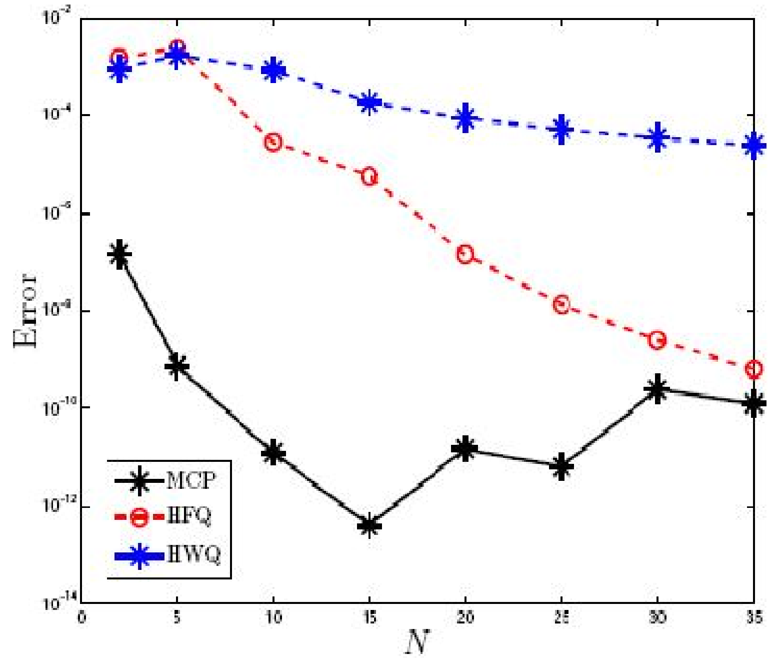

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Figure 2: Absolute error of HFQ, HWQ and GRBF for $\kappa = 1000$

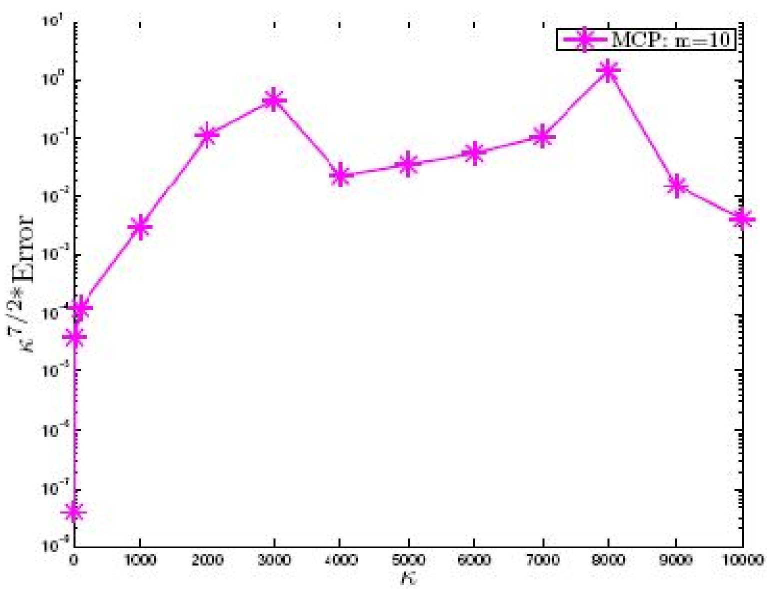

Figure 3: Absolute error scaled by $\kappa^{7/2}$ of GRBF for m = 10

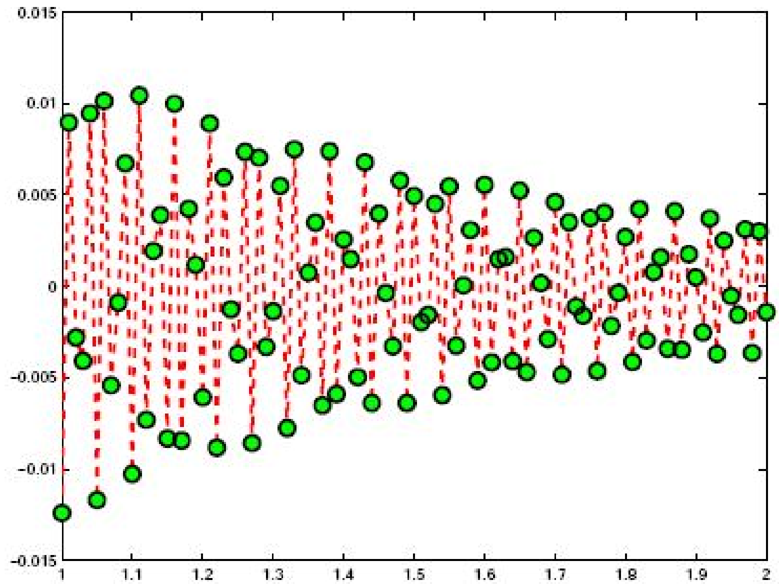

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Figure 4: Oscillatory behaviors of the integrand for $\kappa = 1000$

CONCLUSION

In this paper, a collocation method based on Gaussian RBF and multi-resolution quadrature rules like HFQ and HWQ are used for numerical evaluation of Bessel type of oscillatory integrals. Some theoretical error bounds of the new methods are found. Test problem shows improved results of the new methods.

Nomenclature Box:

Symbols	Description
RBF	Radial basis functions
$\tilde{S}$(x)	Approximate value of S
v	Order of the first kind of Bessel function
c	Shape parameter of the RBF interpolation
c_L, c_U	Lower and upper bounds for optimal value of the shape parameter
$\Upsilon(\kappa, x)$	A vector of Bessel oscillatory functions
O(κ)	Asymptotic order of convergence in terms of Frequency parameter
w_i, s	Unknown coefficients
GRBF	Gaussian RBF based quadrature
HFQ	Hybrid functions based quadrature
HWQ	Haar wavelets based quadrature

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